

# Civil Engineering Systems Analysis

## Lecture XV

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# Sensitivity Analysis

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- ▶ Why do we need sensitivity analysis
  - ▶ The parameters we use are not exact.. They are estimated.. So we need to understand which are the most important parameters, so that we can try to evaluate them carefully i.e. IDENTIFY SENSITIVE PARAMETERS
  - ▶ Also, we would like to know the range of the parameters within which the solution we obtained will remain optimal i.e. RANGE OF THE PARAMETERS
    - ▶ Two kinds: (1) range of parameters that ensures the solution is optimal and (2) range of parameters that ensures the solution has to change
- ▶ It will be extremely time consuming if we had to run everything again
- ▶ So, we use insights from our earlier attempts at solving the simplex to conduct sensitivity analysis

# Sensitivity Analysis

- ▶ We know the kind of algebraic manipulations we do to get our optimal solution

- ▶ 
$$\begin{bmatrix} 1 & \mathbf{c}_B \mathbf{B}^{-1} A - c & \mathbf{c}_B \mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} A & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ X \\ X_S \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} \\ \mathbf{B}^{-1} \mathbf{b} \end{bmatrix}$$

- ▶ We do this for every iteration to arrive at optimal solution

- ▶ Now some parameters in the problem have changed

- ▶  $c \rightarrow \bar{c}$ ,  $b \rightarrow \bar{b}$  and  $A \rightarrow \bar{A}$

- ▶ What we can still do is use the  $\mathbf{B}^{-1}$  we have from above and see what the solution will look like with new parameters

- ▶ I reapply my solution method for the optimal table

- ▶ 
$$\begin{bmatrix} 1 & \mathbf{c}_B \mathbf{B}^{-1} \bar{A} - \bar{c} & \mathbf{c}_B \mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \bar{A} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ X \\ X_S \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B \mathbf{B}^{-1} \bar{\mathbf{b}} \\ \mathbf{B}^{-1} \bar{\mathbf{b}} \end{bmatrix}$$

# Example

Basic	Z	X <sub>A</sub>	X <sub>B</sub>	Y1	Y2	RHS
Z	1	0	0	5/2	1	9460
X <sub>A</sub>	0	1	0	3/4	-5/2	150
X <sub>B</sub>	0	0	1	-1/2	2	220

- ▶ Max  $Z=22X_A+28X_B$
- ▶  $8X_A+10X_B+Y1 =3400$
- ▶  $2X_A+3X_B+Y2 =960$
- ▶  $X_A, X_B, Y1, Y2 \geq 0$
- ▶ Now lets say we changed  $X_B$  parameter to 26
- ▶ The idea is to generate the above table with the new values
- ▶ We are changing only the c vector, so which row will be affected?
- ▶ Only the Z row
- ▶  $c_B B^{-1} = [22 \ 28] \begin{bmatrix} 3/4 & -5/2 \\ -1/2 & 2 \end{bmatrix} = [5/2 \ 1]$
- ▶  $c_B B^{-1} A - c = [5/2 \ 1] \begin{bmatrix} 8 & 10 \\ 2 & 3 \end{bmatrix} - [22 \ 26] = [22 \ 28] - [22 \ 26] = [0 \ 2]$
- ▶  $c_B B^{-1} b = [5/2 \ 1] \begin{bmatrix} 3400 \\ 960 \end{bmatrix} = 9460$
- ▶ Not optimal

Basic	Z	X <sub>A</sub>	X <sub>B</sub>	Y1	Y2	RHS
Z	1	2	0	5/2	1	9460
X <sub>A</sub>	0	1	0	3/4	-5/2	150
X <sub>B</sub>	0	0	1	-1/2	2	220

Update the basic column and continue

# Example

- ▶ Now lets also change 3400 to 3000;  $\mathbf{b} = \begin{bmatrix} 3000 \\ 960 \end{bmatrix}$
- ▶ What else will change?
- ▶ The RHs column
- ▶  $\mathbf{c}_B \mathbf{B}^{-1} \mathbf{b} = [5/2 \quad 1] \begin{bmatrix} 3000 \\ 960 \end{bmatrix} = 8460$
- ▶  $\mathbf{B}^{-1} \mathbf{b} = \begin{bmatrix} 3/4 & -5/2 \\ -1/2 & 2 \end{bmatrix} \begin{bmatrix} 3000 \\ 960 \end{bmatrix} = \begin{bmatrix} -150 \\ 420 \end{bmatrix}$
- ▶ Clearly infeasible
- ▶ How can we make this feasible?

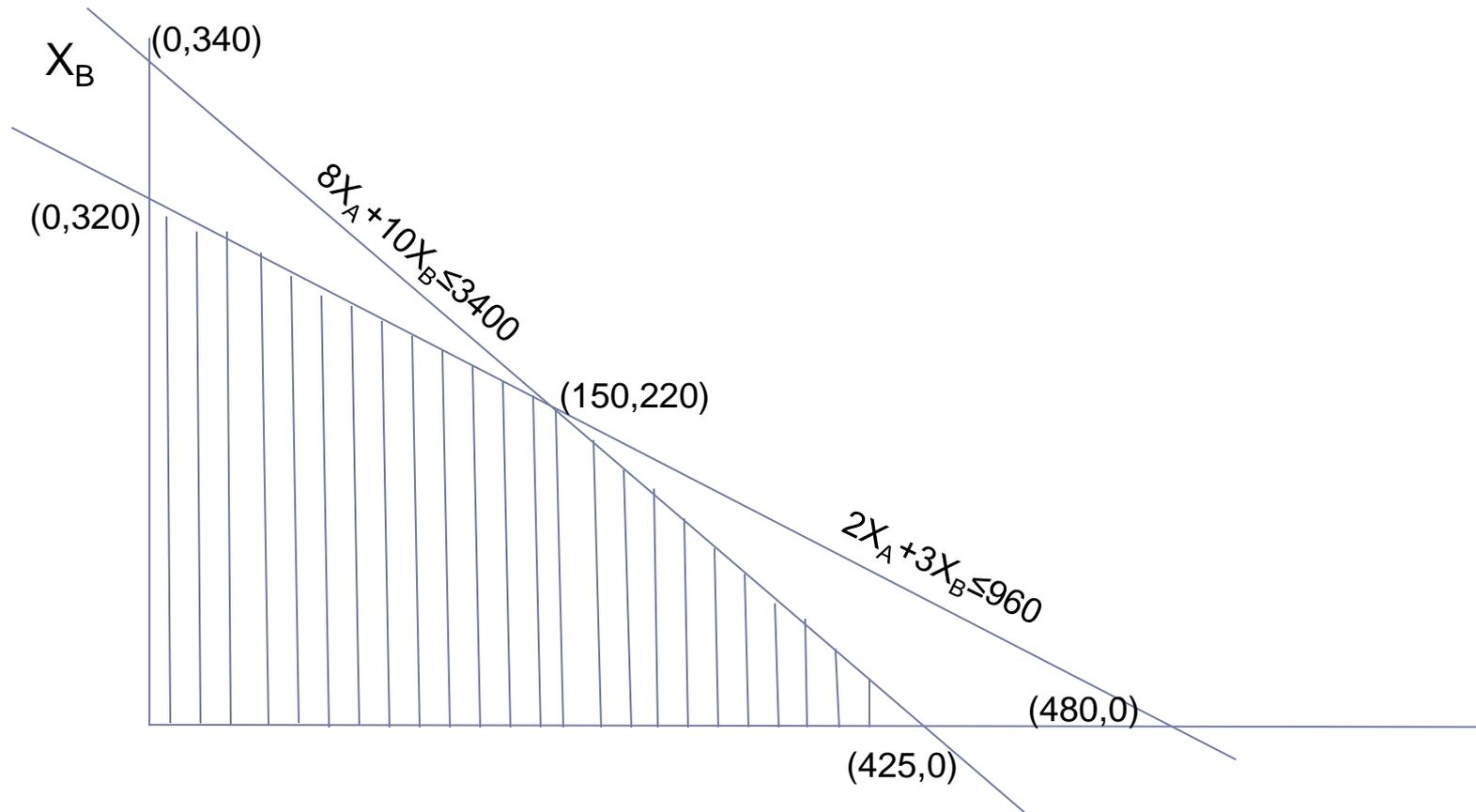
Basic	Z	$X_A$	$X_B$	Y1	Y2	RHS
Z	1	0	2	5/2	1	8460
$X_A$	0	1	0	3/4	-5/2	-150
$X_B$	0	0	1	-1/2	2	420

# How do we address -ive RHS?

- ▶ Some concepts:
- ▶ Max  $Z=22X_A+28X_B$ 
  - ▶  $8X_A+10X_B +u_1=3400$
  - ▶  $2X_A+3X_B + u_2 = 960$
- ▶ Dual
- ▶ Min  $3400y_1+960y_2$ 
  - ▶  $8y_1 + 2y_2 -v_1 = 22$
  - ▶  $10y_1 + 3y_2 - v_2= 28$

Primal		Z=W	Dual	
Basic Solution	Feasible		Basic solution	Feasible
(0,0, 3400, 960)	Yes	0	(0, 0, -22, -28)	No
(0, 320, 200, 0)	Yes	8960	(0,28/3, -10/3, 0 )	No
(425, 0,0,110)	Yes	9350	(11/4, 0, 0, -1/2)	No
(150,220, 0, 0)	Yes	9460	(5/2, 1, 0, 0)	Yes
(0, 340, 0, -60)	No	9520	(14/5, 0, 2/5,0)	Yes
(480, 0, -440, 0)	No	10560	(0, 11, 0, 5)	Yes

# Basic Solutions in a graph



# Dual simplex method

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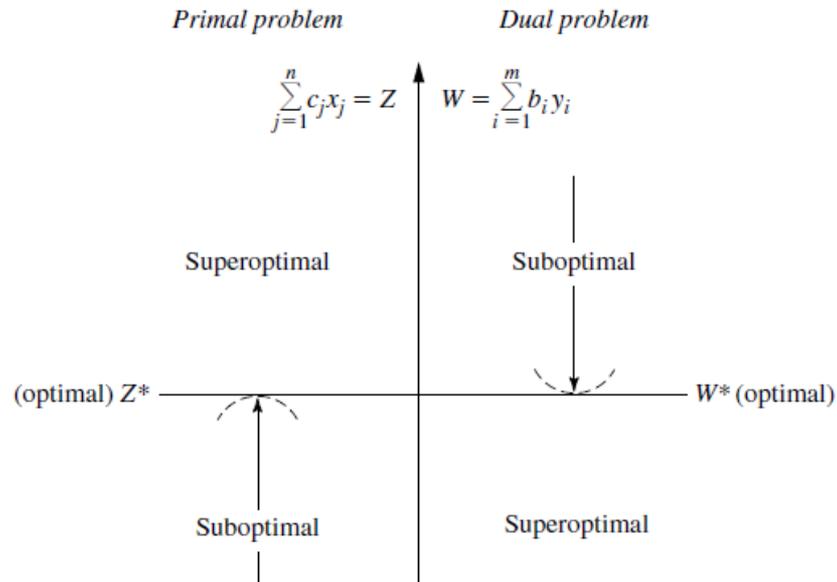
- ▶ What can we say from the Table
- ▶ There is only one solution that is feasible to both primal and dual
- ▶ For others each basic solution is feasible only for one problem
- ▶ For primal (Max) Solutions that are  $<$  optimal are called sub-optimal solutions while those solutions that are  $>$  than optimal are called super-optimal solutions
- ▶ For Dual (Min) Solutions that are  $>$  optimal are called sub-optimal solutions while those solutions that are  $<$  than optimal are called super-optimal solutions

Primal		Z=W	Dual	
Basic Solution	Feasible		Basic solution	Feasible
(0,0, 3400, 960)	Yes	0	(0, 0, -22, -28)	No
(0, 320, 200, 0)	Yes	8960	(0,28/3, -10/3, 0 )	No
(425, 0,0,110)	Yes	9350	(11/4, 0, 0, -1/2)	No
(150,220, 0, 0)	Yes	9460	(5/2, 1, 0, 0)	Yes
(0, 340, 0, -60)	No	9520	(14/5, 0, 2/5,0)	Yes
(480, 0, -440, 0)	No	10560	(0, 11, 0, 5)	Yes

- ▶ The yellow highlighted solutions represent sub-optimal for primal and dual
- ▶ The red highlighted solutions represent super-optimal for primal and dual
- ▶ The green row represents the optimal for primal and dual

# Relationship

- Summarized in these two figures



Primal Basic Solution	Complementary Dual Basic Solution	Both Basic Solutions	
		Primal Feasible?	Dual Feasible?
Suboptimal	Superoptimal	Yes	No
Optimal	Optimal	Yes	Yes
Superoptimal	Suboptimal	No	Yes
Neither feasible nor superoptimal	Neither feasible nor superoptimal	No	No

# Dual Simplex Method

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- ▶ An approach to solving simplex employing dual concepts is labelled as the Dual simplex method
- ▶ Operations are very similar to simplex method
- ▶ It is a mirror image of the simplex method
- ▶ In the simplex method we maintain primal feasibility and arrive at the optimal (we violate dual feasibility in every step except optimal)
- ▶ In the dual simplex we maintain dual feasibility and arrive at the optimal i.e. we violate primal feasibility in every iteration except the final iteration
- ▶ Only change is related to how we choose entering variables and leaving variables
- ▶ Moreover, there is no restriction that RHS has to be positive
  - ▶ So allows us to convert  $\geq$  equations to  $\leq$

# Dual Simplex Method

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- ▶ **Step 1:** Convert all  $\geq$  equations to  $\leq$ , then add slack variables as needed.
  - ▶ Then identify the basis elements such that  $Z_{row}$  elements are 0
- ▶ **Step 2:** Check if all the basic variables are non-negative, if so, we reached optimality
- ▶ **Step 3:** Determine the leaving basic variable:
  - ▶ Select the negative basic variable that has the largest absolute value

# Dual Simplex Method

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- ▶ **Step 4:** Determine the entering basic variable:
  - ▶ Select the nonbasic variable whose coefficient in  $Z$  row reaches zero first as an increasing multiple of the equation containing the leaving basic variable is added to  $Z$  row.
  - ▶ This selection is made by checking the nonbasic variables with *negative coefficients* in that equation (the one containing the leaving basic variable) and selecting the one with the smallest absolute value of the ratio of the  $Z$  row coefficient to the coefficient in that equation
- ▶ **Step 5:** Determine the new basic solution:
  - ▶ Starting from the current set of equations, solve for the basic variables in terms of the nonbasic variables by Gaussian elimination. When we set the nonbasic variables equal to zero, each basic variable (and  $Z$ ) equals the new right-hand side of the one equation in which it appears (with a coefficient of  $+1$ ). Return to Step 2

# Example – Dual Simplex Method

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- ▶ Min  $3400y_1 + 960y_2$ 
  - ▶  $8y_1 + 2y_2 \geq 22$
  - ▶  $10y_1 + 3y_2 \geq 28$
  - ▶  $y_1, y_2 \geq 0$
- ▶ Write this as Max and convert  $\geq$  to  $\leq$
- ▶ Max  $-3400y_1 - 960y_2$ 
  - ▶  $-8y_1 - 2y_2 \leq -22$
  - ▶  $-10y_1 - 3y_2 \leq -28$

Basic	Z	Y1	Y2	V1	V2	Solution
Z	-1	3400	960	0	0	0
V1	0	-8	-2	1	0	-22
V2	0	-10	-3	0	1	-28

# Example

Basic	Z	Y1	Y2	V1	V2	Solution
Z	-1	3400	960	0	0	0
V1	0	-8	-2	1	0	-22
V2	0	-10	-3	0	1	-28

Leaving variable – referred to as pivot row   
Entering variable computed only for -ive coefficients in pivot row:  
 $\text{Min}((3400/10), 960/3) = \text{Y2}$  – entering column

- ▶ Once we know the leaving and entering variables we can employ usual gauss-jordan steps to convert the simplex table for the new basis [v1, y2]

# Example

Basic	Z	Y1	Y2	V1	V2	Solution
Z	-1	200	0	0	320	-8960
V1	0	-4/3	0	1	-2/3	-10/3
Y2	0	10/3	1	0	-1/3	28/3



Leaving variable – referred to as pivot row

Entering variable for -ive coefficients in pivot row:

$\text{Min}((200/4/3), 320/2/3) = Y1$  – entering column

- ▶ Now the new basis is [Y1, Y2]; update the table accordingly

# Example

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Basic	Z	Y1	Y2	V1	V2	Solution
Z	-1	0	0	150	220	-9460
Y1	0	1	0	-3/4	1/2	5/2
Y2	0	0	1	5/2	-1/3	1

All RHS elements positive so optimal solution

- ▶ The tables are similar to what we had for the simplex approach (the dual and primal solns are identical)

# Dual simplex solution

Basic	Z	Y1	Y2	V1	V2	Solution
Z	1	3400	960	0	0	0
V1	0	8	2	1	0	-22
V2	0	10	3	0	1	-28

Basic	Z	Y1	Y2	V1	V2	Solution
Z	1	200	0	0	320	8960
V1	0	-4/3	0	1	-2/3	-10/3
Y2	0	10/3	1	0	-1/3	28/3

Basic	Z	Y1	Y2	V1	V2	Solution
Z	1	0	0	150	220	9460
Y1	0	1	0	-3/4	1/2	5/2
Y2	0	0	1	5/2	-1/3	1

# Simplex solution example

Basic	Z	$X_A$	$X_B$	Y1	Y2	Solution
Z	1	-22	-28	0	0	0
Y1	0	8	10	1	0	3400
Y2	0	2	3	0	1	960

Basic	Z	$X_A$	$X_B$	Y1	Y2	Solution
Z	1	-10/3	0	0	28/3	8960
Y1	0	4/3	0	1	-10/3	200
$X_B$	0	2/3	1	0	1/3	320

Basic	Z	$X_A$	$X_B$	Y1	Y2	Solution
Z	1	0	0	5/2	1	9460
$X_A$	0	1	0	3/4	-5/2	150
$X_B$	0	0	1	-1/2	2	220

# Simplex and Dual Simplex

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## ▶ Differences

- ▶ In simplex we determine the entering variable followed by leaving variable – in Dual Simplex we reverse the steps by determining the leaving variable first and then the entering variable
- ▶ In simplex minimum ratio requires that for the entering variable column the leaving variable candidates should be strictly positive – in Dual Simplex, the entering variable candidates should be strictly negative

## ▶ Similarities

- ▶ Gauss-Jordan operations are same
- ▶ Minimum ratio once computed is employed in the same way

# Other Remarks

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- ▶ Dual Simplex is an effective tool to solve minimization problems
- ▶ It is also very useful for sensitivity analysis – because we can handle negative RHS [Remember we started the dual simplex method as a detour to solving sensitivity analysis]
- ▶ If you noticed, we solved a minimization problem with  $\geq$  constraints without artificial variables -> so quicker because we don't have to push out artificial variable to have a good start point
- ▶ The artificial variable problem in the midterm could have been solved with the dual simplex method. Try that for practice.

# Problem

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▶ Minimize  $12x_1 + 15x_2 + 32x_3 + 26x_4 + 57x_5$

▶ Subject to

$$2x_1 + 3x_2 + 6x_3 + 4x_4 + 8x_5 \geq 9$$

$$3x_1 + 4x_2 + 8x_3 + 6x_4 + 12x_5 \geq 12$$

▶ Max  $-Z = -(12x_1 + 15x_2 + 32x_3 + 26x_4 + 57x_5)$

$$-(2x_1 + 3x_2 + 6x_3 + 4x_4 + 8x_5) \leq -9$$

$$-(3x_1 + 4x_2 + 8x_3 + 6x_4 + 12x_5) \leq -12$$

=>

$$-(2x_1 + 3x_2 + 6x_3 + 4x_4 + 8x_5) + u_1 = -9$$

$$-(3x_1 + 4x_2 + 8x_3 + 6x_4 + 12x_5) + u_2 = -12$$

▶ Write the simplex table and solve it!

# Solution

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Basis	Z	x1	x2	x3	x4	x5	u1	u2	RHS
Z	-1	12	15	32	26	57	0	0	0
u1	0	-2	-3	-6	-4	-8	1	0	-9
u2	0	-3	-4	-8	-6	-12	1	1	-12

- ▶ Leaving variable u2 (highest negative)
- ▶ Check for entering variable
- ▶ Ratio terms  $x1 - 12/3$ ,  $x2 - 15/4$ ,  $x3 - 32/8$ ,  $x4 - 26/6$ ,  $x5 - 57/12$
- ▶ x2 enters
- ▶ New basis u1, x2

# Solution

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Basis	Z	x1	x2	x3	x4	x5	u1	u2	RHS
Z	-1	3/4	0	2	7/2	12	15/4	15/4	-45
u1	0	1/4	0	0	1/2	1	1/4	-3/4	0
x2	0	3/4	1	2	3/2	3	-1/4	-1/4	3

- ▶ No leaving variable
- ▶ Optimal !
- ▶ Easier than having to add artificial variables!

# Back to Sensitivity Analysis

- ▶ Now that we know how to handle the negative RHS we can undertake any form of sensitivity analysis
- ▶ Lets summarize the general procedure for doing sensitivity analysis
- ▶ Step 1: *Revision of model*: Make the desired change or changes in the model to be investigated next.
- ▶ Step 2: *Revision of final tableau using*

$$\begin{bmatrix} 1 & \bar{\mathbf{c}}_B \mathbf{B}^{-1} \bar{\mathbf{A}} - \bar{c} & \bar{\mathbf{c}}_B \mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1} \bar{\mathbf{A}} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathbf{x} \\ \mathbf{x}_S \end{bmatrix} = \begin{bmatrix} \mathbf{c}_B \mathbf{B}^{-1} \bar{\mathbf{b}} \\ \mathbf{B}^{-1} \bar{\mathbf{b}} \end{bmatrix}$$
- ▶ Step 3: *Conversion to proper form from Gaussian elimination*: Convert this tableau to the proper form for identifying and evaluating the current basic solution by applying (as necessary) Gaussian elimination.

# Back to Sensitivity Analysis

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- ▶ Step 4: *Feasibility test*: Test this solution for feasibility by checking whether all its basic variable values in the right-side column of the tableau still are nonnegative.
- ▶ Step 5: *Optimality test*: Test this solution for optimality (if feasible) by checking whether all its nonbasic variable coefficients in row 0 of the tableau still are nonnegative.
- ▶ Step 6: *Reoptimization*: If this solution fails either test, the new optimal solution can be obtained (if desired) by using the current tableau as the initial simplex tableau (and making any necessary conversions) for the simplex method or dual simplex method.

# Specific cases

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- ▶ We have the general procedure to undertake sensitivity analysis
- ▶ However, we can enhance the process based on what parameters are being considered
  - ▶ Case 1: Only  $b$
  - ▶ Case 2: Changes to coefficients of non-basic variables or Introduction of a new variable
  - ▶ Case 3: changes to coefficients of basic variable
  - ▶ Case 4: introduction of new constraint

# Case 1: changes in $b$

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- ▶ If only  $b$  parameters change
  - ▶ Only RHS will change
  - ▶ So two possibilities
    - ▶ We will have a feasible soln (which also will be optimal – no change to  $z_{\text{row}}$ )
    - ▶ We will have a infeasible solution (which can be reoptimized based on dual simplex method)

## Case 2a: changes in the coefficients of a non-basic variable

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- ▶ Lets say the coefficients corresponding to the  $j^{\text{th}}$  element have changed (in  $c$  vector and the  $A_j$ )
- ▶ What will change in the optimal table?
  - ▶ The column corresponding to the  $j^{\text{th}}$  element: (1) Zrow element  $c_B B^{-1} \bar{A}_j - \bar{c}_j$ , (2) coefficient of  $x_j$  in rows 1 through  $m$  is  $B^{-1} \bar{A}_j$
- ▶ If Zrow element is negative then we have to reoptimize else nothing needed (even no need to compute  $B^{-1} \bar{A}_j$ )
- ▶ Please remember if we are to only check the optimality we can use dual!
- ▶ But to reoptimize we need the sensitivity analysis approach

## Case 2b: introduction of a new variable

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- ▶ Lets say after we have solved for optimality we might realize we have not considered a variable in the model
- ▶ Any easy solution to address this?
- ▶ We can treat it exactly in a similar way as the Case 2a.
- ▶ We can do this by pretending the values of coefficients for the variable were 0 in the optimal table and now they have changed to whatever we want to introduce
- ▶ So same approach will allow us to address this

# References

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- ▶ Hillier F.S and G. J. Lieberman. Introduction to Operations Research, Ninth Edition, McGraw-Hill, 2010