Civil Engineering Systems Analysis Lecture XII

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Today's Learning Objectives

- Mid-term
- Dual

Linear Programming Dual Problem

Points to Remember

- "=" constraint gives an unrestricted variable and vice versa
- Desirable constraints yield desirable variables and vice-versa
  - In a minimization ≥ is desirable hence the variable corresponding to that will be ≥ 0 (look at constraints 2 and 4 in primal)
  - In a minimization ≤ is undesirable and hence it gives us ≤0 variable (look at constraint 3 in primal)
- In any problem non-negativity is desirable,
  - non-negative variable yield desirable constraint signs (see x1 in primal)
  - Negative variables yield non-desirable constraint signs (see x3 in primal

Primal Dual relationships

- Weak duality theorem
- If x is a feasible solution for the primal problem and y is a feasible solution to the dual problem, then $cx \le yb$
- Lets look at an example
- Primal:
  - Max $Z = 6x1+5x2$
  - $x1+x2 \le 5$
  - $3x1+2x2 \le 12$
  - $x1, x2 \ge 0$

(0,0) Z = 0
(3,1) Z = 23

- Dual
  - Min $W = 5y1+12y2$
  - $y1+ 3y2 \ge 6$
  - $y1 + 2y2 \ge 5$
  - $y1, y2 \ge 0$

(3,3) Z = 51
(1,2) Z = 29

Primal Dual relationships

- Weak duality – mathematically
- Lets say we have a feasible soln to primal and dual
- Then we write
- $W = 5y1+12y2$
- $W \geq (x1+x2)y1+ (3x1+2x2)y2$
- $W \geq x1(y1+3y2)+ x2(y1+2y2)$
- $W \geq 6x1+ 5x2 = Z$
- $W \geq Z$
- If we generalize this to matrices, we proved the weak duality

Primal Dual relationships

- Strong – duality theorem
- If x^* is an optimal solution for the primal problem and y^* is an optimal solution for the dual problem then $cx^* = y^*b$
- If $cx^* = y^*b$ then x^* and y^* are solutions to primal and dual respectively
- Primal:
  - Max $Z = 6x1+5x2$
  - $x1+x2 \leq 5$
  - $3x1+2x2 \leq 12$
  - $x1, x2 \geq 0$
  - Try (2, 3)

Dual:

Min $W = 5y1+12y2$

$y1+ 3y2 \geq 6$

$y1 + 2y2 \geq 5$

$y1, y2 \geq 0$

Try (3,1)

Primal Dual Relationship

- Why do we care about weak duality and strong duality?

- We care because when we want to develop computer models to solve for large problems, we might use heuristics. Heuristics tap into such logical rules and try to solve the problem quickly
- For example in a large problem we have access to primal and dual
- Now if I can quickly find solutions to primal and dual with same objective function value.. I know I hit my optimum..
  - I can write my code with this as my criterion.. a search based algorithm that does not worry about simplex and so on!

Primal Dual relationships

- If one problem has a feasible solution and bounded objective function then so does the other problem
- If one problem is feasible with an unbounded objective function then the other problem has no feasible solution
- If one problem has no feasible solution, then other problem either has no feasible solution or is unbounded

- Again, these relationships can be exploited in solving large problems
- Lets say I figure out that the Dual has unbounded solution (1-2 iterations).. I don't have to solve primal, I know it is infeasible
- Once I find out for example, dual is infeasible then I can check directly for unboundedness or infeasibility in primal

Primal Dual relationships

- Complementary slackness conditions
- If x* and y* are optimal solutions to primal and dual respectively, we have xV + yU=0
- U and V vector of slack variables for primal and dual
- We can write this as $\sum x_i V_i + \sum y_j U_j = 0$
- Because all of these ≥ 0
- We can write $\sum x_i V_i = 0$ and $\sum y_j U_j = 0$
- And in fact, $x_i V_i = 0$; $y_j U_j = 0$

- What does this mean?
  - If a particular primal variable is basic then it corresponding dual slack will be 0

Complementary slackness

- Primal
  - Maximize $Z = 3x1+5x2$
    - $x1 \leq 4$
    - $2x2 \leq 12$
    - $3x1 + 2x2 \leq 18$
    - $x1, x2 \geq 0$
  - Maximize $Z = 3x1+5x2$
    - $x1 + u1 = 4$
    - $2x2 + u2 = 12$
    - $3x1 + 2x2 + u3 = 18$
    - $x1, x2, u1, u2, u3 \geq 0$
- Solution
  - (2, 6, 2, 0, 0)

- Dual
  - Minimize $W = 4y1+12y2+18y3$
  - $y1 + \quad 3y3 \geq 3$
  - $2y2+2y3 \geq 5$
  - $y1, y2, y3 \geq 0$
  - Minimize $W = 4y1+12y2+18y3$
  - $y1 + \quad 3y3 - v1 = 3$
  - $2y2+2y3 - v2 = 5$
  - $y1, y2, y3, v1, v2 \geq 0$

Solution

From complementary slackness

x1 is basic v1 =0; x2 is basic v2 =0

u1 is basic so y1 =0; u2 =0, y2 non-zero; u3 =0, y3 non-zero

Solving we get Y2 = 3/2, Y3 =1! Z = 36

Complementary Slackness

- Primal:
  - Max Z = 6x1+5x2
    - x1+x2 + u1 = 5
    - 3x1+2x2 + u2 =12
    - x1, x2, u1, u2 ≥ 0
  - Optimal Soln (2,3,0,0) and Z = 27
- Does this satisfy complementary slackness
- Yes..
- x1*v1=0, x2*v2=0
- y1*u1=0, y2*u2=0

- Dual
  - Min 5y1 + 12y2
    - y1 + 3y2 –v1 = 6
    - y1 + 2y2 – v2 = 5
    - y1, y2, v1, v2 ≥ 0
  - Optimal soln (3,1,0,0) and Z = 27

Dual Mathematical interpretation

- Maximize 22 X_A+28X_B
  - $8X_A+10X_B \le 3400$
  - $2X_A+3X_B \le 960$
  - $X_A, X_B \ge 0$
- Optimal solution is (150, 220, 0, 0)
- Dual
- Min 3400y1+960y2
  - 8y1+2y2 ≥ 22
  - 10y1+3y2 ≥ 28
  - y1,y2 ≥0
- Dual solution from complementary slackness
  - u1 = 0; y1 non-zero
  - u2 = 0; y2 non-zero
  - x1 basic, v1 = 0
  - x2 basic, v2=0
  - Soln (5/2, 1, 0, 0)
- The dual represents the marginal value for the resource

Dual – Marginal value interpretation

- Maximize 22 X_A+28X_B
  - $8X_A+10X_B \leq 3400$
  - $2X_A+3X_B \leq 960$
  - $X_A, X_B \geq 0$
- Lets say 3400 is increased by δ
- What will happen to the optimal solution
- Lets assume X_A and X_B are still in the basis
- In that case
- Solve for X_A and X_B;
- $X_B = 220 - \delta/2$; $X_A = 150 + 3\delta/4$
- Znew = $9460+5/2\delta$
- We can do the same for 960
- Simplex provides the entire range of values

Dual from Simplex table

- Solution to the problem in Simplex

Basic	Z	X_A	X_B	Y1	Y2	Solution
Z	1	0	0	5/2	1	9460
X_A	0	1	0	3/4	-5/2	150
X_B	0	0	1	-1/2	2	220

- By observing the z row we can determine the dual solution at optimality

Z	**1**	**0**	**0**	**5/2**	**1**	**9460**
		v1	v2	y1	y2	

Min 3400y1+960y2
8y1+2y2 ≥ 22;
10y1+3y2 ≥ 28; y1,y2 ≥0

Full set of iterations

Basic	Z	X_A	X_B	Y1	Y2	Solution
Z	1	-22	-28	0	0	0
Y1	0	8	10	1	0	3400
Y2	0	2	3	0	1	960

Basic	Z	X_A	X_B	Y1	Y2	Solution
Z	1	-10/3	0	0	28/3	8960
Y1	0	4/3	0	1	-10/3	200
X_B	0	2/3	1	0	1/3	320

Basic	Z	X_A	X_B	Y1	Y2	Solution
Z	1	0	0	5/2	1	9460
X_A	0	1	0	3/4	-5/2	150
X_B	0	0	1	-1/2	2	220

Points to note

- Whenever the problem is feasible for the primal, but not yet optimal, the dual is infeasible
- But at optimality dual becomes feasible (as well as optimal)
- We will exploit this relationship in sensitivity analysis

Sensitivity Analysis

- Dual provides the shadow prices
  - We know that we have this information in the simplex itself
- Dual allows us to do sensitivity analysis more efficiently
- If I change something in the primal, the change will also transfer to the dual also.
- Based on which problem will give me the result efficiently I can use that to address the sensitivity

Changes to a non-basic variable

- You have solved the primal to optimality and found your basic variables
  - This means these are the set of variables that affect your bottom line
- Now, you realize the coefficients of the non-basic variable are not reliable. So how do we test that?
- What is the impact of changes to the basic feasible solution?
  - Is the solution still feasible?
    - Yes
  - Is the solution still optimal?
    - We need to check

Example

- Max $x1+4x2+2x3$
- $x1+ x2+ x3 \leq 6$
- $x1+ 3x2+ 2x3 \leq 12$
- $x1+ 2x2 \leq 6$
- $x1, x2, x3 \geq 0$
- Optimal solution is (0, 3, 1.5)

Example –simplex solution

	x1	x2	x3	s1	s2	s3	RHS
Z	-1	-4	-2	0	0	0	0
S1	1	1	1	1	0	0	6
S2	1	3	2	0	1	0	12
S3	1	2	0	0	0	1	6

	x1	x2	x3	s1	s2	s3	RHS
Z	1	0	-2	0	0	2	12
S1	1/2	0	1	1	0	- 1/2	3
S2	- 1/2	0	2	0	1	-3/2	3
x2	1/2	1	0	0	0	1/2	3

	x1	x2	x3	s1	s2	s3	RHS
Z	1/2	0	0	0	1	1/2	15
S1	¾	0	0	1	- 1/2	1/4	3/2
x3	- ¼	0	1	0	1/2	- 3/4	3/2
x2	1/2	1	0	0	0	1/2	3

Example

- Now lets say the coefficient of x1 in obj. function is 2
- Is the solution still feasible?
  - Yes
- What about optimality?
  - Is it easy to figure that out?
  - This is where the dual comes in handy
- We know that since the solution is optimal, the corresponding dual solution is feasible
- Lets quickly compute the dual

Example

- Dual
- Min 6y1+12y2+6y3
- y1+y2+1y3≥1
- y1+3y2+2y3≥4
- y1+2y2≥2
- y1, y2, y3 ≥0
- Can we determine the value of y1, y2, y3from simplex table
- (0,1, 0.5, 0.5, 0, 0)

Example

- Now when we changed the coefficient in obj. function for primal what will change in the dual
  - The change will be only in the constraint 1
  - Now we know that the optimal solution to primal should remain feasible for the dual
  - So, lets see if the solution is still feasible with new constraint ($y1+y2+1y3\geq 2$)
    - Soln 0, 1, 0.5 => not feasible
    - Hence when we change the coefficient of x1 in objective function optimal basic solution will change!
    - We checked it with just one constraint
- A good tool to make quick checks

Primal Dual relationships

- Consider the case of adding a new variable to the problem

- For example, a company produces three products currently and are considering producing a new product

- What will this mean?

Example

- Max x1+4x2+2x3
  - x1+ x2+ x3 ≤ 6
  - x1+ 3x2+ 2x3 ≤ 12
  - x1+ 2x2 ≤ 6
  - x1, x2, x3≥0
- Optimal solution is (0, 3, 1.5)
- Now the company is interested to check if x_{new} will provide more profit
- New Problem
- Max $x1+4x2+2x3+3x_{new}$
  - $x1+ x2+ x3 +x_{new} \leq 6$
  - $x1+ 3x2+ 2x3+2x_{new} \leq 12$
  - $x1+ 2x2 +3x_{new} \leq 6$
  - $x1, x2, x3, x_{new} \geq 0$

Example

- Is the solution still feasible?
  - Yes
- What about optimality?
  - Check dual feasibility
- We know that since the solution is optimal, the corresponding dual solution is feasible

Example

- New Dual
- Min 6y1+12y2+6y3
  - y1+y2+1y3≥1
  - y1+3y2+2y3≥4
  - y1+2y2 ≥2
  - y1+2y2+3y3≥3 (NEW)
  - y1, y2, y3 ≥0
- Current optimal solution (0,1, 0.5)
- Do I have to worry about the entire dual?
- Only one constraint corresponding to x_{new} is enough!
- Now check feasibility for new constraint -> still feasible -> so the new product does not change the optimal solution for the company, so you do not start producing x_{new}!

Sensitivity Analysis

- Why do we need sensitivity analysis
  - The parameters we use are not exact.. They are estimated.. So we need to understand which are the most important parameters, so that we can try to evaluate them carefully i.e. IDENTIFY SENSITIVE PARAMETERS
  - Also, we would like to know the range of the parameters within which the solution we obtained will remain optimal i.e. RANGE OF THE PARAMETERS
    - Two kinds: (1) range of parameters that ensures the solution is optimal and (2) range of parameters that ensures the solution has to change
- It will be extremely time consuming if we had to run everything again
- So, we use insights from our earlier attempts at solving the simplex to conduct sensitivity analysis

Sensitivity Analysis

- We know the kind of algebraic manipulations we do to get our optimal solution
- $$\begin{bmatrix} 1 & \mathbf{c_B B}^{-1}A - c & \mathbf{c_B B}^{-1} \\ 0 & \mathbf{B}^{-1}A & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathrm{x} \\ \mathrm{x_s} \end{bmatrix} = \begin{bmatrix} \mathbf{c_B B}^{-1}\mathbf{b} \\ \mathbf{B}^{-1}\mathbf{b} \end{bmatrix}$$
- We do this for every iteration to arrive at optimal solution
- Now some parameters in the problem have changed
- c -> $\bar{c}$, b -> $\bar{\mathrm{b}}$ and A-> $\bar{\mathrm{A}}$
- What we can still do is use the B^{-1} we have from above and see what the solution will look like with new parameters
- I reapply my solution method for the optimal table
- $$\begin{bmatrix} 1 & \overline{\mathbf{c_B}}\mathbf{B}^{-1}\bar{\mathrm{A}} - \bar{c} & \overline{\mathbf{c_B}}\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1}\bar{\mathrm{A}} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathrm{x} \\ \mathrm{x_s} \end{bmatrix} = \begin{bmatrix} \mathbf{c_B B}^{-1}\bar{\mathrm{b}} \\ \mathbf{B}^{-1}\bar{\mathrm{b}} \end{bmatrix}$$

Example

Basic	Z	X_A	X_B	Y1	Y2	RHS
Z	1	0	0	5/2	1	9460
X_A	0	1	0	3/4	-5/2	150
X_B	0	0	1	-1/2	2	220

- Max $Z=22X_A+28X_B$
- $8X_A+10X_B+Y1=3400$
- $2X_A+3X_B+Y2=960$
- $X_A,X_B,Y1,Y2\geq 0$
- Now lets say we changed X_B parameter to 26
- The idea is to generate the above table with the new values
- We are changing only the c vector, so which row will be affected?
- Only the Z row
- $c_B B^{-1} = [22\ 26]\begin{bmatrix} 3/4 & -5/2 \\ -1/2 & 2 \end{bmatrix} = [7/2\ \ -3\ ]$
- $c_B B^{-1}A\text{-}c = [7/2\ \ -3\ ]\begin{bmatrix} 8 & 10 \\ 2 & 3 \end{bmatrix}$-[22 26] =[22 26] – [22 26] = [0 0]
- $c_B B^{-1}b = [7/2\ \ -3\ ]\begin{bmatrix} 3400 \\ 960 \end{bmatrix} = 9020$
- Not optimal

Basic	Z	X_A	X_B	Y1	Y2	RHS
Z	1	0	0	14/4	-3	9020
X_A	0	1	0	3/4	-5/2	150
X_B	0	0	1	-1/2	2	220

Continue the iterations till optimality!

Example

- Now lets also change 3400 to 3000; b = $\begin{bmatrix} 3000 \\ 960 \end{bmatrix}$
- What else will change?
- The RHs column
- $c_B B^{-1} b = [7/2 \quad -3\,] \begin{bmatrix} 3000 \\ 960 \end{bmatrix} = 7620$
- $B^{-1} b = \begin{bmatrix} 3/4 & -5/2 \\ -1/2 & 2 \end{bmatrix} \begin{bmatrix} 3000 \\ 960 \end{bmatrix} = \begin{bmatrix} -150 \\ 420 \end{bmatrix}$
- Clearly infeasible
- How can we make this feasible?

Basic	Z	X_A	X_B	Y1	Y2	RHS
Z	1	0	0	14/4	-3	7620
X_A	0	1	0	3/4	-5/2	-150
X_B	0	0	1	-1/2	2	420

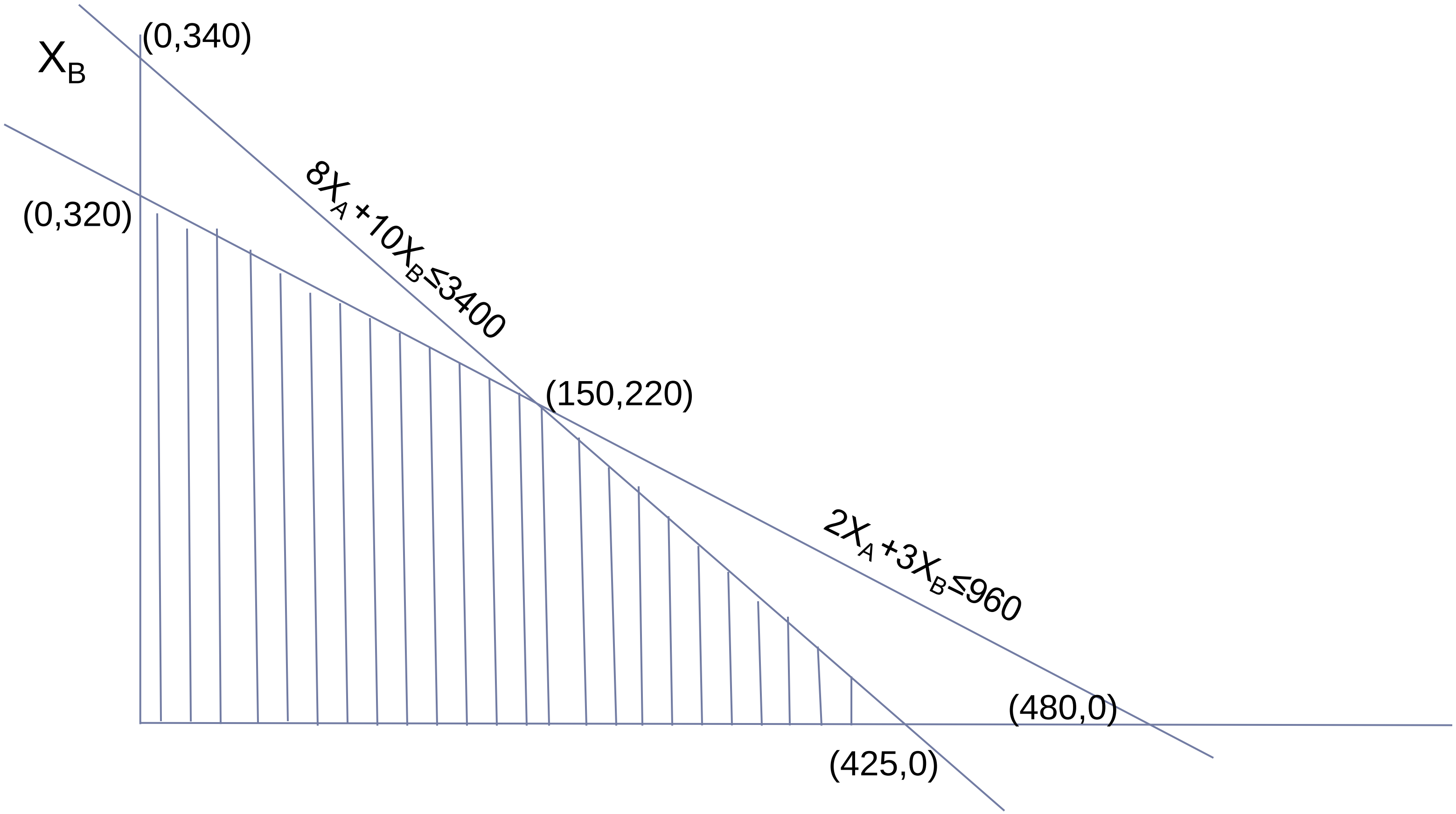
How do we address –ive RHS?

- Some concepts:
- Max $Z=22X_A+28X_B$
  - $8X_A+10X_B+u1=3400$
  - $2X_A+3X_B + u2 = 960$

- Dual
- Min $3400y1+960y2$
  - $8y1 + 2y2 - v1 = 22$
  - $10y1 + 3y2 - v2= 28$

Primal		Z=W	Dual	
Basic Solution	Feasible		Basic solution	Feasible
(0,0, 3400, 960)	Yes	0	(0, 0, -22, -28)	No
(0, 320, 200, 0)	Yes	8960	(0,28/3, -10/3, 0)	No
(425, 0,0,110)	Yes	9350	(11/4, 0, 0, -1/2)	No
(150,220, 0, 0)	Yes	9460	(5/2, 1, 0, 0)	Yes
(0, 340, 0, -60)	No	9520	(14/5, 0, 2/5,0)	Yes
(480, 0, -440, 0)	No	10560	(0, 11, 0, 5)	Yes

Basic Solutions in a graph

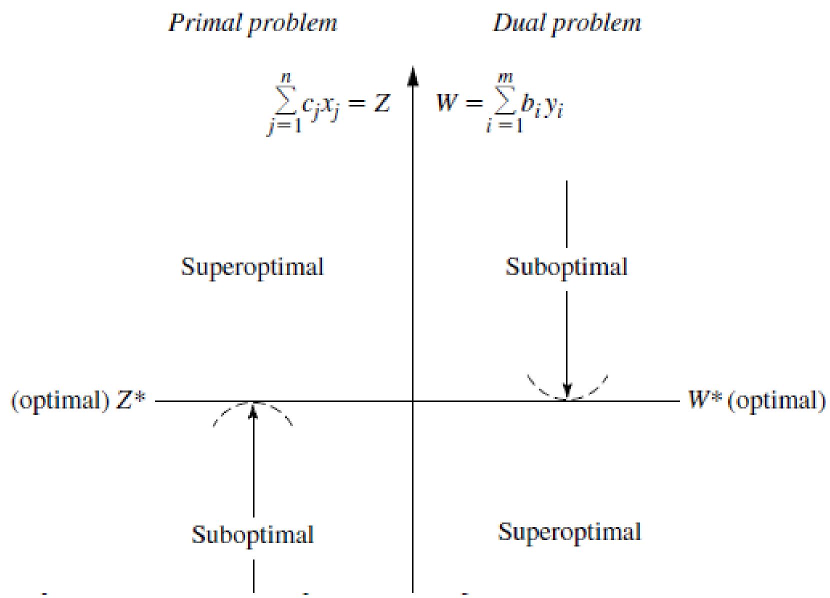


Dual simplex method

- What can we say from the Table
- There is only one solution that is feasible to both primal and dual
- For others each basic solution is feasible only for one problem
- For primal (Max) Solutions that are < optimal are called sub-optimal solutions while those solutions that are > than optimal are called super-optimal solutions
- For Dual (Min) Solutions that are > optimal are called sub-optimal solutions while those solutions that are < than optimal are called super-optimal solutions

Relationship

- Summarized in these two figures



Primal Basic Solution	**Complementary Dual Basic Solution**	**Both Basic Solutions**	
		Primal Feasible?	**Dual Feasible?**
Suboptimal	Superoptimal	Yes	No
Optimal	Optimal	Yes	Yes
Superoptimal	Suboptimal	No	Yes
Neither feasible nor superoptimal	Neither feasible nor superoptimal	No	No

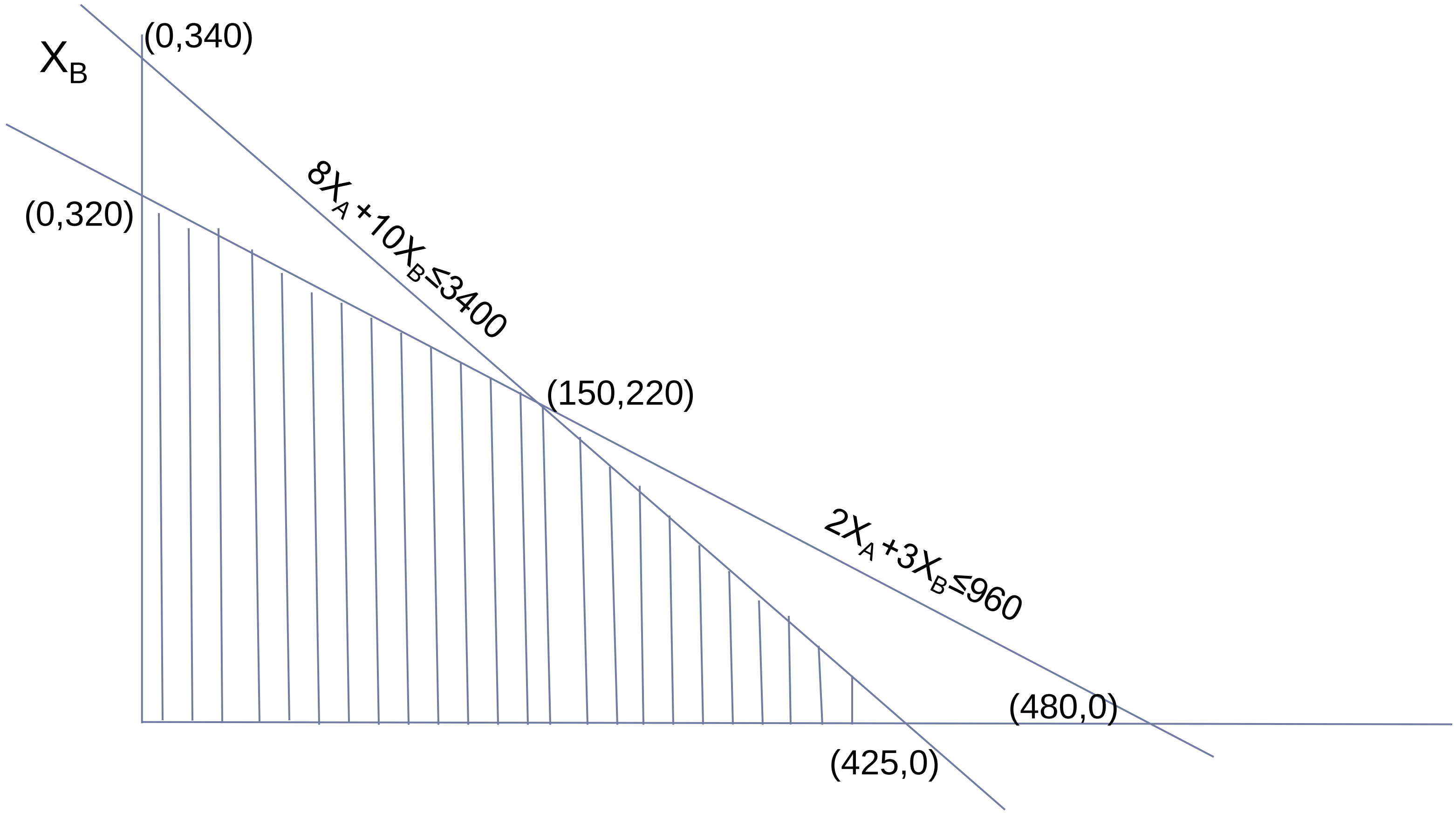
How do we address –ive RHS?

- Some concepts:
- Max $Z=22X_A+28X_B$
  - $8X_A+10X_B+u1=3400$
  - $2X_A+3X_B + u2 = 960$

- Dual
- Min 3400y1+960y2
  - 8y1 + 2y2 -v1 = 22
  - 10y1 + 3y2 - v2= 28

Primal		Z=W	Dual	
Basic Solution	Feasible		Basic solution	Feasible
(0,0, 3400, 960)	Yes	0	(0, 0, -22, -28)	No
(0, 320, 200, 0)	Yes	8960	(0,28/3, -10/3, 0)	No
(425, 0,0,110)	Yes	9350	(11/4, 0, 0, -1/2)	No
(150,220, 0, 0)	Yes	9460	(5/2, 1, 0, 0)	Yes
(0, 340, 0, -60)	No	9520	(14/5, 0, 2/5,0)	Yes
(480, 0, -440, 0)	No	10560	(0, 11, 0, 5)	Yes

Basic Solutions in a graph

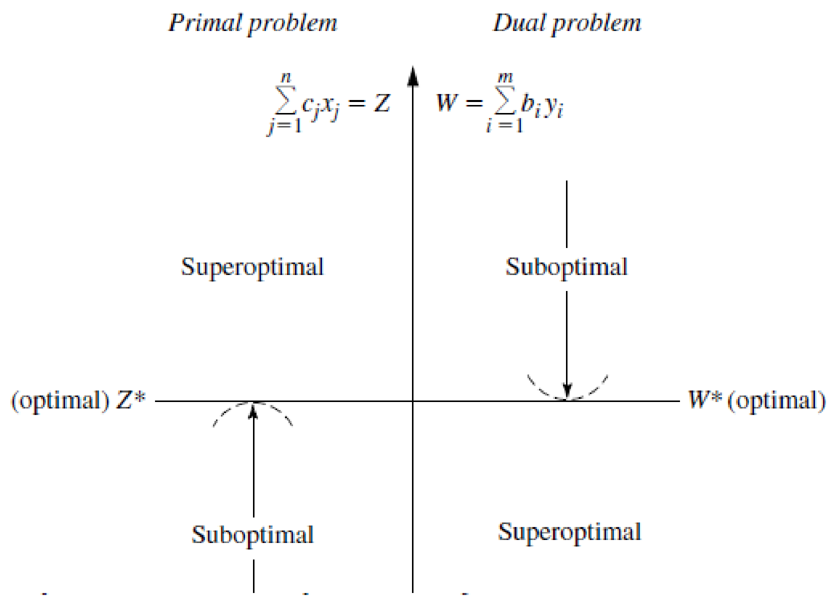


Dual simplex method

- What can we say from the Table
- There is only one solution that is feasible to both primal and dual
- For others each basic solution is feasible only for one problem
- For primal (Max) Solutions that are < optimal are called sub-optimal solutions while those solutions that are > than optimal are called super-optimal solutions
- For Dual (Min) Solutions that are > optimal are called sub-optimal solutions while those solutions that are < than optimal are called super-optimal solutions

Relationship

- Summarized in these two figures



Primal Basic Solution	Complementary Dual Basic Solution	Both Basic Solutions: Primal Feasible?	Both Basic Solutions: Dual Feasible?
Suboptimal	Superoptimal	Yes	No
Optimal	Optimal	Yes	Yes
Superoptimal	Suboptimal	No	Yes
Neither feasible nor superoptimal	Neither feasible nor superoptimal	No	No

Dual Simplex Method

- An approach to solving simplex employing dual concepts is labelled as the Dual simplex method
- Operations are very similar to simplex method
- It is a mirror image of the simplex method
- In the simplex method we maintain primal feasibility and arrive at the optimal (we violate dual feasibility in every step except optimal)
- In the dual simplex we maintain dual feasibility and arrive at the optimal i.e. we violate primal feasibility in every iteration except the final iteration
- Only change is related to how we choose entering variables and leaving variables
- Moreover, there is no restriction that RHS has to be positive
  - So allows us to convert ≥ equations to ≤

Simplex solution example

Basic	Z	X_A	X_B	Y1	Y2	Solution
Z	1	-22	-28	0	0	0
Y1	0	8	10	1	0	3400
Y2	0	2	3	0	1	960

Basic	Z	X_A	X_B	Y1	Y2	Solution
Z	1	-10/3	0	0	28/3	8960
Y1	0	4/3	0	1	-10/3	200
X_B	0	2/3	1	0	1/3	320

Basic	Z	X_A	X_B	Y1	Y2	Solution
Z	1	0	0	5/2	1	9460
X_A	0	1	0	3/4	-5/2	150
X_B	0	0	1	-1/2	2	220

Method

- **Step 1:** Convert all ≥ equations to ≤, then add slack variables as needed. Now identify the basis elements such that Zrow elements are 0
- **Step 2:** Check if all the basic variables are non-negative, if so, we reached optimality
- **Step 3:** Determine the leaving basic variable: select the negative basic variable that has the largest absolute value
- **Step 4:** Determine the entering basic variable: Select the nonbasic variable whose coefficient in Zrow reaches zero first as an increasing multiple of the equation containing the leaving basic variable is added to Zrow. This selection is made by checking the nonbasic variables with *negative coefficients* in that equation (the one containing the leaving basic variable) and selecting the one with the smallest absolute value of the ratio of the Zrow coefficient to the coefficient in that equation
- **Step 5:** Determine the new basic solution: Starting from the current set of equations, solve for the basic variables in terms of the nonbasic variables by Gaussian elimination. When we set the nonbasic variables equal to zero, each basic variable (and Z) equals the new right-hand side of the one equation in which it appears (with a coefficient of +1). Return to Step 2

Example

- Min $3400y1+960y2$
  - $8y1 + 2y2 \geq 22$
  - $10y1 + 3y2 \geq 28$
  - $y1, y2 \geq 0$
- Write this as Max
- Max $-3400y1 - 960y2$
  - $-8y1 - 2y2 \leq -22$
  - $-10y1 - 3y2 \leq -28$

Basic	Z	Y1	Y2	V1	V2	Solution
Z	1	3400	960	0	0	0
V1	0	-8	-2	1	0	-22
V2	0	-10	-3	0	1	-28

Example

Basic	Z	Y1	Y2	V1	V2	Solution
Z	1	3400	960	0	0	0
V1	0	-8	-2	1	0	-22
V2	0	-10	-3	0	1	-28

Leaving variable – referred to as pivot row

Entering variable for –ive coefficients in pivot row:
Min((3400/10), 960/3) = Y2 – entering column

Basic	Z	Y1	Y2	V1	V2	Solution
Z	1	200	0	0	320	-8960
V1	0	-4/3	0	1	-2/3	-10/3
Y2	0	10/3	1	0	-1/3	28/3

Leaving variable – referred to as pivot row

Entering variable for –ive coefficients in pivot row:
Min((200/4/3), 320/2/3) = Y1 – entering column

Example

Basic	Z	Y1	Y2	V1	V2	Solution
Z	1	0	0	150	220	-9460
Y1	0	1	0	-3/4	1/2	5/2
Y2	0	0	1	5/2	-1/3	1

All RHS elements positive so optimal solution

- The tables are similar to what we had for the simplex approach (the dual and primal solns are identical)

Dual simplex solution

Basic	Z	Y1	Y2	V1	V2	Solution
Z	1	3400	960	0	0	0
V1	0	8	2	1	0	-22
V2	0	10	3	0	1	-28

Basic	Z	Y1	Y2	V1	V2	Solution
Z	1	200	0	0	320	8960
V1	0	-4/3	0	1	-2/3	-10/3
Y2	0	10/3	1	0	-1/3	28/3

Basic	Z	Y1	Y2	V1	V2	Solution
Z	1	0	0	150	220	9460
Y1	0	1	0	-3/4	1/2	5/2
Y2	0	0	1	5/2	-1/3	1

Remarks

- If you noticed, we solved a minimization problem with ≥ constraints without artificial variables -> so quicker because we don't have to push out artificial variable to have a good start point

- If you have not realized the 6^{th} problem in the midterm could have been solved with the dual simplex method. Try that for practice.

- Remember we started the dual simplex method as a detour to solving sensitivity analysis

- Now that we know how to handle negative RHS we can continue looking at sensitivity analysis

Back to Sensitivity Analysis

- Now that we know how to handle the negative RHS we can undertake any form of sensitivity analysis
- Lets summarize the general procedure for doing sensitivity analysis
- Step 1: *Revision of model:* Make the desired change or changes in the model to be investigated next.
- Step 2: *Revision of final tableau using*

$$\begin{bmatrix} 1 & \overline{\mathbf{c_B}}\mathbf{B}^{-1}\overline{\mathrm{A}} - \bar{c} & \overline{\mathbf{c_B}}\mathbf{B}^{-1} \\ 0 & \mathbf{B}^{-1}\overline{\mathrm{A}} & \mathbf{B}^{-1} \end{bmatrix} \begin{bmatrix} Z \\ \mathrm{x} \\ \mathrm{x_s} \end{bmatrix} = \begin{bmatrix} \mathbf{c_B}\mathbf{B}^{-1}\bar{\mathrm{b}} \\ \mathbf{B}^{-1}\bar{\mathrm{b}} \end{bmatrix}$$

- Step 3: *Conversion to proper form from Gaussian elimination:* Convert this tableau to the proper form for identifying and evaluating the current basic solution by applying (as necessary) Gaussian elimination.

Back to Sensitivity Analysis

- Step 4: *Feasibility test:* Test this solution for feasibility by checking whether all its basic variable values in the right-side column of the tableau still are nonnegative.
- Step 5: *Optimality test:* Test this solution for optimality (if feasible) by checking whether all its nonbasic variable coefficients in row 0 of the tableau still are nonnegative.
- Step 6: *Reoptimization:* If this solution fails either test, the new optimal solution can be obtained (if desired) by using the current tableau as the initial simplex tableau (and making any necessary conversions) for the simplex method or dual simplex method.

Specific cases

- We have the general procedure to undertake sensitivity analysis
- However, we can enhance the process based on what parameters are being considered
  - Case 1: Only b
  - Case 2: Changes to coefficients of non-basic variables or Introduction of a new variable
  - Case 3: changes to coefficients of basic variable
  - Case 4: introduction of new constraint

Case 1: changes in b

- If only b parameters change
  - Only RHS will change
  - So two possibilities
    - We will have a feasible soln (which also will be optimal – no change to zrow)
    - We will have a infeasible solution (which can be reoptimized based on dual simplex method)

Case 2a: changes in the coefficients f a non-basic variable

- Lets say the coefficients corresponding to the j^{th} element have changed (in c vector and the A_j)
- What will change in the optimal table?
  - The column corresponding to the j^{th} element: (1) Zrow element $c_B B^{-1}\bar{A}_j - \bar{c}_j$, (2) coefficient of xj in rows 1 through m is $B^{-1}\bar{A}_j$
- If Zrow element is negative then we have to reoptimize else nothing needed (even no need to compute $B^{-1}\bar{A}_j$)

- Please remember if we are to only check the optimality we can use dual!
- But to reoptimize we need the sensitivity analysis approach

Case 2b: introduction of a new variable

- Lets say after we have solved for optimality we might realize we have not considered a variable in the model
- Any easy solution to address this?
- We can treat it exactly in a similar way as the Case 2a.
- We can do this by pretending the values of coefficients for the variable were 0 in the optimal table and now they have changed to whatever we want to introduce
- So same approach will allow us to address this

Case 3: changes to coefficients of basic variable

- Slightly more complicated because when we change the coefficients of the basic variable, after applying the correction to the table for corresponding elements ($\overline{c_B} B^{-1}\overline{A_j} - \overline{c_j}$ and $B^{-1}\overline{A_j}$)
- Now these elements might not have the standard identity for the constraints and 0 for the zrow .. so we reapply Gauss-Jordan transformations to obtain the standard form
- Now when we do these changes the solution might violate optimality and/or feasibility
- Then we just reoptimize using simplex or dual simplex

Case 4: Introduction of new constraint

- Lets say we overlooked a particular constraint for the problem then?
- The solution to this is easy
- Check if the solution remains feasible to the new constraint.. If it does it will also be optimal
- The reason being the constraint can only remove a feasible solution (i.e. cause a reduction in obj. func) never increase it.. So if its feasible we are good
- If the constraint is infeasible .. Add it as a new row in the simplex table (with the slack or artificial variable of this constraint as basic element) , change it appropriately to satisfy basic row criterion and reoptimize if needed.

Remarks

- All 4 cases are special cases of the general approach
- We are just trying to undertake sensitivity analysis efficiently
- We can always go back to general procedure if the problem does not fall into these 4 cases
- The tutorial sessions will cover numerical examples for the sensitivity analysis examples as well as dual simplex method

Systematic Sensitivity analysis

- So far, we looked at the influence of changes to parameters on the optimal solutions

- Now we will look at parameterizing the change

- For example, consider increasing the resource b_i by an amount θ. In that case we can compute the change in the solution as a function of θ and predict the changes based on the range of θ

Consider an example

- Max Z = 3x1+5x2
- Subject to
  - x1≤4
  - 2x2 ≤12
  - 3x1+2x2 ≤18
  - x1≥0,x2 ≥0
- Lets say resources 2 (12) has become 12+θ

	x1	x2	x3	x4	x5	Solution
Z	0	0	0	3/2	1	36
x3	0	0	1	1/3	-1/3	2
x2	0	1	0	½	0	6
x1	1	0	0	-1/3	1/3	2

Sensitivity analysis

- We already know the increase in Z value (How?)
  - The dual value gives you that information (3/2)
- But what about optimal x values
- We can compute $B^{-1}b$ and $C_B B^{-1}b$
- B^{-1} is available from the optimal table = $\begin{matrix} 1 & 1/3 & -1/3 \\ 0 & 1/2 & 0 \\ 0 & -1/3 & 1/3 \end{matrix}$
- $B^{-1}b = \begin{matrix} 2 + 1/3\theta \\ 6 + 1/2\theta \\ 2 - 1/3\theta \end{matrix}$
- Now can we make any comments on the range of the solution i.e. what value of θ will change the optimal solution (if it will!)
- When $2-1/3\theta < 0$ we will have a change in the optimal solution => $\theta > 6$ will change the solution
- Test this!

Sensitivity analysis

- The entire sensitivity analysis discussion can be repeated for the parametric method i.e. instead of changing the parameters change them by θ
- This method provides us the range over which the current solution remains optimal
- The change can be attributed to multiple variables
- Sometimes, it is possible to have connected changes i.e. By increasing the profit on one good you reduce the profit on the second
- How can we handle such changes

Sensitivity analysis

- For the previous example, lets say increase in b2 reduces b3 twice as fast
- b2 -> b2+ θ; b3-> b3-2θ
- In the example, b2 = 12+θ, b3 = 18- 2θ
- Now redo the sensitivity analysis. In this case the range you will get for the solution will be different from the range you obtained from the previous example
- The same changes can happen in C vector and A Matrix. The approach is still the same (use B^{-1})
- In tutorial and homework I will give you examples of such problems

References

- Hillier F.S and G. J. Lieberman. Introduction to Operations Research, Ninth Edition, McGraw-Hill, 2010